

Vertex algebras and elliptic genus

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Vadim Schechtman

This is a report on a joint work with V. Gorbounov, F. Malikov and A. Vaintrob, cf. [MSV], [MS1], [MS2], [GMS].

1. Let X be a smooth complex variety (algebraic or analytic). One can associate with X a canonically defined gerbe (champ of groupoids) \mathfrak{D}_X^{ch} called *the gerbe of chiral differential operators* on X . For an open $U \subset X$, the objects of the groupoid $\mathfrak{D}_X(U)$ are sheaves of vertex algebras over X , called *sheaves of chiral differential operators* on X . The gerbe \mathfrak{D}_X is bounded by the lien isomorphic to the sheaf $\Omega_X^{2,cl}$ of closed two-forms.

The equivalence class

$$c(\mathfrak{D}_X) \in H^2(X, \Omega_X^{2,cl})$$

is equal to

$$2ch_2(\mathcal{T}_X) := c_1^2(\mathcal{T}_X) - 2c_2(\mathcal{T}_X) \quad (1.1)$$

where $c_i(\mathcal{T}_X) \in H^i(X, \Omega_X^{i,cl})$ are the Chern classes.

For example, (1.1) vanishes for products of curves, or for flag spaces G/B .

2. Let E be a vector bundle over X . Generalizing the previous construction, one can define the gerbe $\mathfrak{D}_{\Lambda E}$ of chiral differential operators on the exterior algebra ΛE , also bounded by the sheaf $\Omega_X^{2,cl}$.

Its equivalence class is equal to

$$c(\mathfrak{D}_{\Lambda E}) = 2ch_2(\mathcal{T}_X) - 2ch_2(E) \quad (2.1)$$

For example, (2.1) vanishes for $E = \Omega_X^1$ or \mathcal{T}_X .

3. Consider the case $E = \Omega_X^1$. Let us denote the gerbe $\mathfrak{D}_{\Lambda E}$ of chiral differential operators on the de Rham algebra of differential forms by \mathfrak{D}_Ω . The groupoid $\mathfrak{D}_\Omega(X)$ is non-empty.

In fact, one can define certain *canonical* object \mathcal{D}_Ω of $\mathfrak{D}_\Omega(X)$. It was introduced in [MSV], and called *chiral de Rham complex* there. Recently A. Beilinson gave a nice coordinate-free construction of \mathcal{D}_Ω and some of its generalizations in the language of chiral algebras (in the sense of Beilinson - Drinfeld).

\mathcal{D}_Ω is a sheaf of conformal vertex superalgebras, \mathbb{Z} -graded by *fermionic charge*. For a compact X , the cohomology algebra

$$H^*(X, \mathcal{D}_\Omega) = \bigoplus_{p=0}^n \bigoplus_{q=-\infty}^{\infty} \bigoplus_{i=0}^{\infty} H^p(X, \mathcal{D}_{\Omega,i}^q) \quad (3.1)$$

is called the *chiral Hodge cohomology* of X and has some nice properties. Here $\mathcal{D}_{\Omega,i}^q$ denotes the component of conformal weight i and fermionic charge q , $n = \dim(X)$.

The space (3.1) is a conformal vertex superalgebra. It is a $N = 2$ superalgebra if X is Calabi-Yau (i.e. when the canonical bundle is trivial). We have the "Poincaré duality":

$$H^p(X, \mathcal{D}_{\Omega, i}^q)^* = H^{n-p}(X, \mathcal{D}_{\Omega, i}^{n-q}) \quad (3.2)$$

As was noted in [BL], if X is Calabi-Yau then the generating series

$$\chi_X(q, y) := y^{n/2} \sum_{a, b, i} (-1)^{a+b} \dim H^a(X, \mathcal{D}_{\Omega, i}^b) y^b q^i \quad (3.3)$$

coincides with the Ochanine — Witten elliptic genus of X .

References

[BL] L. Borisov, A. Libgober, Elliptic genera and applications to Mirror symmetry, math/9906126.

[GMS] V. Gorbounov, F. Malikov, V. Schechtman, Gerbes of chiral differential operators, math.AG/9906117.

[MSV] F. Malikov, V. Schechtman, A. Vaintrob, Chiral de Rham complex, *Comm. Math. Phys.* (1999), to appear; math.AG/9803041.

[MS1] F. Malikov, V. Schechtman, Chiral de Rham complex. II, *D.B. Fuchs' 60-th Anniversary volume* (1999), to appear; math.AG/9901065.

[MS2] F. Malikov, V. Schechtman, Chiral Poincaré duality, math.AG/9905008.

Department of Mathematics, University of Glasgow, 15 University Gardens,
Glasgow G12 8QW, UK; vs@maths.gla.ac.uk, vadik@mpim-bonn.mpg.de